NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2768

SUPERSONIC FLOW WITH WHIRL AND VORTICITY IN AXISYMMETRIC CHANNELS

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SUMMARY

The axially symmetric supersonic steady flow is treated for a non-viscous fluid flowing with whirl component between two arbitrary coaxial surfaces of revolution. The equations that describe this motion are expressed in characteristic coordinates and in this form are used to determine the meridional (axial-radial) velocities for arbitrary distributions of the tangential velocity components.

Solution of the supersonic characteristic equations is investigated for arbitrary axisymmetric flow fields with and without vorticity. This solution includes cases in which the channel shape and the inlet velocity distribution are prescribed. Also included are cases in which one of the channel surfaces, the desired velocities on that surface (within certain limitations), and the inlet velocity distribution are prescribed.

INTRODUCTION

A survey of the literature showed that a method of design or analysis for supersonic flow with whirl in the three-dimensional region of an axially symmetric channel formed by two coaxial surfaces of revolution without blades is not available. In the past, axially symmetric channels of this type have been designed from one-dimensional flow theory or the stream-filament method, which does not consider the character of the flow (subsonic or supersonic). More recently, a method of solution was outlined in reference 1 for subsonic flow by means of an iteration process using orthogonal curvilinear coordinates for arbitrary known boundaries.

A method is described herein which was devised at the NACA Lewis laboratory for analyzing supersonic flow in this type channel. This method can be used to obtain the velocities in the flow field when the boundaries are known with either prescribed rotational or irrotational flow at the inlet. It can also be used to obtain the opposite boundary to satisfy a prescribed boundary with the desired velocities on that boundary (within certain regions which are defined later) for either a prescribed rotational or irrotational flow at the inlet or the outlet.

The velocity components of the flow equations are treated directly in accordance with the procedure outlined in reference 2. The same technique for axisymmetric flow in rotating impellers is utilized in reference 3.

In addition to describing the method in a mathematical outline, this report gives numerical examples which demonstrate some of the phenomena present in the flow in either supersonic vaneless difusers or the channel between the outlet of a supersonic compressor impeller and an annular cascade of supersonic diffuser blades. The method is particularly suited to determine pressure gradients and possible shock configurations when supersonic flow exists in radial diffusers and in annular channels at the inlet or outlet of a compressor or turbine. This allows the determination of possible flow separation within the channel and suggests design modifications to eliminate the losses resulting from shocks.

DERIVATION OF FLOW EQUATIONS

The flow equations to be developed consider the steady flow of a nonviscous compressible fluid under conditions of axial symmetry and isentropic state changes. The coordinates used and the velocity components chosen are shown in figure 1.

Stream Function

With the use of the symbols defined in the appendix, the continuity equation for steady flow is

$$\nabla \cdot (\rho \overline{V}) = 0$$

In cylindrical coordinates, for axial symmetry (partial derivations of the fluid properties with respect to θ and zero), this equation becomes

$$\frac{\partial}{\partial r}(\rho v r) + \frac{\partial}{\partial z}(\rho w r) = 0 \tag{1}$$

Equation (1) implies the existence of a stream function (of two variables r and z) such that

$$\left.\begin{aligned}\frac{\partial\psi}{\partial r} &= \rho w r \\ \frac{\partial\psi}{\partial z} &= -\rho v r \\ \frac{\partial\psi}{\partial z} &= 0\end{aligned}\right\} \qquad (2)$$

2499 In vector notation, equations (2) are

$$\nabla\psi = \rho r j \times \overline{V} \qquad (2a)$$

Equation of Motion

For steady nonviscous flow, the equation of motion is

$$(\overline{V}\cdot\nabla)\ \overline{V} = -\frac{\nabla p}{\rho}$$

For isentropic conditions, the pressure term may be eliminated since

$$\frac{\nabla p}{\rho} = \nabla H$$

Hence,

$$(\overline{V}\cdot\nabla)\ \overline{V} = -\nabla H$$

The general energy equation for the conditions of no work done and no heat transfer states that the stagnation or total energy of the fluid

$$H_T \equiv H + \frac{V^2}{2}$$

is constant on any streamline. If the gradient is taken, this equation becomes

$$\nabla H_T = \nabla H + \nabla \frac{V^2}{2}$$

Upon substitution into the equation of motion,

$$(\overline{V}\cdot\nabla)\ \overline{V} - \nabla \frac{V^2}{2} = -\nabla H_T$$

or

$$\overline{V} \times (\nabla \times \overline{V}) = \nabla H_T \tag{3}$$

which is one form of the equation of motion; for later development into the characteristic equations, however, it is convenient to transform the equation in the following manner: If the scalar product of equation (3) with the unit vector j is taken and the fact that for axial symmetry

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$$j \cdot \nabla H_T = 0 \tag{4}$$

is considered, the resulting equation is

$$\overline{V} \cdot \left[j \times (\nabla \times \overline{V}) \right] = 0 \tag{5}$$

This equation further reduces to

$$\overline{V} \cdot \nabla (ru) = 0 \tag{6}$$

since for axial symmetry,

$$j \times (\nabla \times V) = \frac{1}{r} \nabla (ru) \tag{7}$$

From equations (2a), (3), (4), (6), and (7), it is evident that $\nabla(ru)$, ∇H_T, and $\nabla\psi$ are normal to the velocity vector $\overline{V}$ and the unit vector j. Therefore, ∇H_T and $\nabla(ru)$ must be parallel to $\nabla\psi$ and hence H_T and (ru) must be functions of ψ only (reference 4); that is,

$$\nabla H_T = \nabla\psi \frac{dH_T}{d\psi} \tag{8}$$

$$\nabla (ru) = \nabla\psi \frac{d(ru)}{d\psi} \tag{9}$$

For any vector $\overline{A}$ and any unit vector n,

$$\overline{A} = n\,(n \cdot \overline{A}) - n \times (n \times \overline{A})$$

may be obtained by expansion of the product $n \times n \times \overline{A}$. Therefore the curl $\nabla \times \overline{V}$ can be written

$$\nabla \times \overline{V} = j\left[j \cdot (\nabla \times \overline{V})\right] - j \times \left[j \times (\nabla \times \overline{V})\right]$$

Inserting this expression for the curl into equation (3), expanding the triple vector product, and utilizing equation (5) give the following equation:

$$\overline{V} \times j\left[j \cdot (\nabla \times \overline{V})\right] + (j \cdot \overline{V})\left[j \times (\nabla \times \overline{V})\right] = \nabla H_T$$

When equations (2a), (7), (8), and (9) are inserted, this equation becomes

$$\nabla \psi \left[\frac{j \cdot (\nabla \times \overline{V})}{\rho r} - \frac{u}{r}\frac{d(ru)}{d\psi} + \frac{dH_T}{d\psi}\right] = 0 \tag{10}$$

which is the most convenient vector form of the equation of motion.

General Equations in Cylindrical Coordinates

The equations for the flow have been developed, thus far, in vector notation. By the use of cylindrical coordinates, the equations are translated to scalar notation.

Continuity equation. - If the vector form of the continuity equation (1) is expanded and divided by ρ, the following equation is obtained:

$$\nabla \cdot \overline{V} + \overline{V} \cdot \frac{\nabla \rho}{\rho} = 0$$

By use of the isentropic flow relation

$$\frac{\nabla \rho}{\rho} = \frac{\nabla H}{a^2}$$

the energy equation, and the fact that $\overline{V} \cdot \nabla H_T = 0$, there results

$$\nabla \cdot \overline{V} - \frac{1}{a^2}\overline{V} \cdot \nabla \frac{V^2}{2} = 0$$

In cylindrical coordinates, for axial symmetry, this expression becomes

$$\left(1 - \frac{v^2}{a^2}\right) v_r + \left(1 - \frac{w^2}{a^2}\right) w_z - \frac{vw}{a^2} (w_r + v_z) - \left(\frac{vu}{a^2}\right) u_r - \left(\frac{wu}{a^2}\right) u_z + \frac{v}{r} = 0$$

From equation (6), the u_r and u_z terms may be eliminated and the continuity equation in cylindrical coordinates becomes 2499

$$\left(1 - \frac{v^2}{a^2}\right) v_r + \left(1 - \frac{w^2}{a^2}\right) w_z - \frac{vw}{a^2} (w_r + v_z) + \left(1 + \frac{u^2}{a^2}\right) \frac{v}{r} = 0 \qquad (11)$$

Equation of motion. - From equation (10), since ψ is not constant, the scalar equation of motion becomes

$$\rho u \frac{d(ru)}{d\psi} \rho r \frac{dH_T}{d\psi} - (v_z - w_r) = 0 \qquad (12)$$

wherein $j \cdot (\nabla \times \overline{V})$ has been expressed in cylindrical coordinates for axial symmetry. Equation (12) is applicable for inlet flow conditions which have tangential velocity and nonuniform energy level. If the energy level of the flow entering is uniform, the term $dH_T/d\psi$ vanishes, corresponding to the vector equation $\overline{V} \times (\nabla \times \overline{V}) = 0$, which was solved in reference 1 for subsonic flow.

If in the flow entering (ru) is a constant or zero, the term $d(ru)/d\psi$ vanishes. If the flow entering is isoenergetic $(dH_T/d\psi = 0)$ and (ru) is constant or zero, the flow is irrotational and equation (12) reduces to the familiar expression for two-dimensional flow,

$$\frac{\partial v}{\partial z} = \frac{\partial w}{\partial r}$$

Characteristic Equations

The supersonic flow under consideration is governed by equations (11) and (12), which form a system of two equations for the two variables w and v as functions of both r and z. This system of equations can be solved by the method outlined in reference 2 and leads to two basic characteristic equations having new independent variables ξ and η, instead of r and z. Each of these two basic characteristic equations contains derivatives with respect to only one of the independent variables ξ or η, instead of both independent variables

as in the case of equations (11) and (12). The independent variables ξ and η are the characteristic parameters. Two additional equations are also obtained and these are used to find the position coordinates r and z; each of these two equations contains derivatives of r and z with respect to only one of the independent variables ξ or η. The characteristic lines have slopes dr/dz. The slope of the characteristic line designated by ζ_+ corresponds to variable ξ and constant η, and the slope designated ζ_- corresponds to constant ξ and variable η.
2499 The results in applying this method are shown in the following equations in which ζ represents either of the slopes of the characteristic lines and σ represents either of the coordinates ξ or η:

$$r_\sigma = \zeta z_\sigma \tag{13}$$

and

$$w_\sigma + X v_\sigma + Y z_\sigma = 0 \tag{14}$$

where

$$\zeta = \frac{1}{1 - \frac{w^2}{a^2}} \left(- \frac{wv}{a^2} \pm \sqrt{M_m^2 - 1} \right) \tag{15}$$

$$X = \frac{1}{1 - \frac{w^2}{a^2}} \left(- \frac{wv}{a^2} \mp \sqrt{M_m^2 - 1} \right) \tag{16}$$

$$Y = \frac{1}{1 - \frac{w^2}{a^2}} \left\{ \pm \sqrt{M_m^2 - 1} \left[\rho u \frac{d(ru)}{d\psi} - \rho r \frac{dH_T}{d\psi} \right] + \left(1 + \frac{u}{a^2} \right) \frac{v}{r} \right\} \tag{17}$$

In the terms in which a choice of sign exists in equations (15) to (17), the upper sign refers to a coefficient with a plus subscript and the lower sign to a coefficient with a minus subscript. Thus, X_+ and Y_+ correspond to variable ξ and constant η, and X_- and Y_- correspond to constant ξ and variable η. Examination of equations (15) to (17) shows that $\zeta_+ = X_-$ and $\zeta_- = X_+$ and that the system is hyperbolic if

$M_m^2 > 1$. As shown in reference 2, hodograph solutions tabulated in advance of application to the solution of any problem that may arise cannot be made, since equation (14) is nonhomogeneous $(Y \neq 0)$ and cannot be integrated directly. As a result, the four equations obtained from equations (13) and (14), when σ is equal to ξ or η, must be integrated simultaneously in a step-by-step procedure.

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Although the following equations will not be used in this report, they are included to show a method of solution when the flow is subsonic. Combining equations (2) and (12) gives

$$\psi_{rr} + \psi_{zz} - \frac{\partial \ln(\rho r)}{\partial r} \psi_r - \frac{\partial \ln(\rho r)}{\partial z} \psi_z + (\rho r)^2 \left[\frac{u}{r} \frac{d(ru)}{\partial \psi} - \frac{dH_T}{\partial \psi}\right] = 0 \tag{18}$$

which is the principal equation for solution of subsonic flow. This equation may be shown to be of the elliptic type if the meridional velocity $V_m = \sqrt{v^2 + w^2}$ is less than the speed of sound. The subsonic flow problem may be solved by a system of equations consisting of equations (2), (18), and the density relation for isentropic flow

$$\rho = \rho_{T,i} \left(1 - \frac{V^2}{2\, H_{T,i}}\right)^{\frac{1}{\gamma-1}} \tag{19}$$

This system of equations can be solved for given boundary conditions by relaxation methods (references 5 and 6) to give the distribution of ψ in the meridional plane, from which the distribution of the velocities may be obtained from equation (2). The application of relaxation methods to a similar system of equations for the subsonic flow through radial- and mixed-flow centrifugal compressors is shown in reference 7. Another method of solution for a similar system of equations is discussed in detail in reference 8 for flow through turbomachines with arbitrary hub and casing shapes.

NUMERICAL PROCEDURE

Two types of problem exist, the direct or analysis problem in which the velocities are determined for given boundaries of an arbitrary channel shape, and the inverse or design problem in which the boundaries are determined for prescribed velocities. Both the supersonic analysis

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and the design problems may be solved by the characteristic equations and will be discussed in detail after a consideration of the method of computation. In general, suppose that conditions are known at P_1 and P_2 (fig. 2) and that it is desired to determine conditions at P_3. Equation (13) written in difference form for these points is

$$r_3 - r_1 = (\zeta_+)_{1-3} (z_3 - z_1) \tag{20}$$

$$r_3 - r_2 = (\zeta_-)_{2-3} (z_3 - z_2) \tag{21}$$

where $(\zeta_+)_{1-3}$ indicates the average value of ζ_+ between points P_1 and P_3, and $(\zeta_-)_{2-3}$ indicates the average value of ζ_- between points P_2 and P_3. Similarly, equations (14) may be written in difference form:

$$w_3 - w_1 + (X_+)_{1-3} (v_3 - v_1) + (Y_+)_{1-3} (z_3 - z_1) = 0 \tag{22}$$

$$w_3 - w_2 + (X_-)_{2-3} (v_3 - v_2) + (Y_-)_{2-3} (z_3 - z_2) = 0 \tag{23}$$

These equations also indicate that the coefficients are averages of the values at the appropriate points. Equations (20) and (21) may be solved for r_3 and z_3, after which equations (22) and (23) may be solved for w_3 and v_3. For the first approximation, only the values of the coefficients ζ, X, Y are known at points P_1 and P_2, and these values are used in place of averages. After the computation of approximate values of the position coordinates and the velocity components at P_3, the coefficients ζ, X, Y may be computed at this point. Average coefficients may now be found and a better approximation to the position coordinates and the velocity components obtained at P_3 may be obtained. Convergence to an answer of satisfactory accuracy, for a given interval or net size, is obtained by repeating this process as often as required. The answers will converge to the solution with fewer iterations, however, the smaller the interval between points. The system of equations (20) to (23) may also be applied to points P_1, P_2', and P_3 if the known points happen to have their positions related to each other in this manner instead of in the manner indicated by P_1, P_2, and P_3. In order to determine $d(ru)/d\psi$ and $dH_T/d\psi$ when they are not equal to zero, in each approximation obtained in the preceding process, ψ_3 must be computed and can be obtained from

$$\psi_3 - \psi = \int_z^{z_3} (\psi_r \zeta + \psi_z)\, dz = \int_r^{r_3} \left(\psi_r + \frac{\psi_z}{\zeta}\right) dr \tag{24}$$

In this equation, ζ_+ is used if the integration is performed along the constant-η line, or ζ_-, if along the constant-ξ line. Since the derivatives $d(ru)/d\psi$ and $dH_T/d\psi$ are total derivatives, that is, ru and H_T are functions of ψ only, these derivatives can be evaluated from the inlet flow conditions for any streamline (ψ-function). Hence from the value of ψ_3, either $d(ru)/d\psi$, or $dH_T/d\psi$, or both may be found at P_3. 2499

Suppose that conditions are known at P_2 (fig. 3) and it is desired to determine conditions at the intersection P_3 of the constant-ξ line which passes through P_2 and the known boundary AB. The boundary AB is a known function and may be specified as

$$r = g(z)$$

In difference form, equation (13) written for these points along a constant-ξ line is equation (21) and since

$$r_3 = g_3(z) \tag{25}$$

equations (21) and (25) may be solved for r_3 and z_3. The slope of the boundary at point P_3 is given by

$$\left(\frac{dg}{dz}\right)_3 = \left(\frac{dr}{dz}\right)_3 = \frac{v_3}{w_3} \tag{26}$$

and since equation (23) for these points along a curve of constant ξ is the difference form of equation (14), equations (23) and (26) may be solved for v_3 and w_3. As previously mentioned, values at the known point are used in lieu of averages for the first approximation and the process is repeated as often as necessary to obtain the desired accuracy. Conditions at point P_3' are determined in a similar manner. If either $d(ru)/d\psi \neq 0$, or $dH_T/d\psi \neq 0$, or both, these conditions can be readily evaluated, since ψ is known at the boundaries.

Analysis Problem

The application of the characteristic equations to the analysis problem is quite simple and straightforward. In this problem the boundary curves are known, and the flow conditions are usually specified at the inlet to the axisymmetric channel. A step-by-step solution along characteristic lines is then made, following the method of computation previously discussed, until the velocities throughout the field or channel have been obtained. From interpolation along the characteristic lines, the velocities across the outlet may then be obtained. In lieu of specified flow conditions at the inlet, the flow conditions may be specified at the outlet and a solution obtained for the flow in the channel and across the inlet. If, in a particular problem being solved, $d(ru)/d\psi = 0$ and $dH_T/d\psi = 0$, integration to obtain the stream function along a characteristic line can be omitted if the streamlines are not desired; even in this case, however, concurrent integration of equation (24) as the problem is being computed is recommended, since comparison with the known values of the stream function at the boundaries furnishes an excellent check.

Design Problem

In applying the characteristic equations to obtain a solution to one type of design problem, all inlet conditions and part of the outlet conditions are specified and the velocities are prescribed where it is possible along a given wall, in order to obtain a solution for the opposite wall and the channel velocities. In the previous discussion in which figure 2 was considered, it was implied that the boundary conditions are known on a curve C containing such points as P_1 and P_2 and that the solution is then extended into a region which contains such points as P_3 and in which $ru(\psi)$ and $H_T(\psi)$ are known. On curve C, since w, v, ru, ρ_T, and H_T are known, all the other variables, such as $d(ru)/d\psi$, $dH_T/d\psi$, ψ, u, ρ, and a^2, may be computed. Curve C should correspond to the line AB in figure 4 on which the inlet flow conditions are specified or to the given boundary curve BCDE. If the inlet flow conditions are specified, however, the flow is completely determined in the region bounded by the inlet AB, the portion of the given boundary curve BC, and the characteristic curve ξ_1, and the velocities may not be prescribed along BC. Conversely, if it is desired to prescribe the velocities completely along the curve BCDE, then only one of the velocity components v or w may be prescribed along AB. For the prescribed outlet conditions, the same considerations are valid in the region DEF. If the values of (ru) and H_T

are specified functions of ψ, various combinations of the remaining variables can be specified along the boundaries for the design problem as shown in the following table (see fig. 4):

Case	Given boundary			Inlet	Outlet
	BC	CD	DE	AB	EF
1	w,v	w,v	w,v	w or v or v/w	w or v or v/w
2	v,w	w,v	w,v	w,v	w or v or v/w
3	w,v	w,v	v/w	w or v or v/w	w,v
4	v,w	w,v	v/w	w,v	w,v

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In case the inlet boundary falls along a curve AB, where B is between C and D, the inlet section AB has the same type of boundary conditions as the outer contour BD; that is, the conditions on AB and BD are independent of each other. Similar relations apply at the outlet for an outlet line DE, when E is between C and D. A more general discussion of the relation between boundary shapes and boundary conditions can be found in reference 3.

After the boundary conditions have been specified, a step-by-step solution is made along the characteristic lines until the velocities throughout the channel have been determined. Integration of the velocities in the channel along characteristic lines gives the stream function ψ. Points of equal ψ corresponding to the full mass flow yield the final streamline or the required opposite boundary.

NUMERICAL EXAMPLES

The characteristic equations were applied and solutions were obtained for three examples of supersonic flow with whirl through axisymmetric channels. These examples embody the analysis problem for irrotational inlet flow and the design problem for both irrotational and rotational inlet flow. The purpose of these solutions is to present some typical problems in order to illustrate the application of the characteristic equations. For all examples, the inlet flow is isoenergetic, that is, $dH_T/d\psi = 0$. The dimensions r and z are made dimensionless by the inlet radius of the outer contour in all examples. Also, the velocities, the static pressures, and the densities are made dimensionless by the inlet stagnation conditions of the velocity of sound, the pressure, and the density, respectively. The stream function ψ is made dimensionless by the value of ψ for the full mass flow. Dimensionless quantities are indicated by the superscript $*$. The value of the ratio of specific heats γ was taken equal to 1.400 for all examples.

Example I

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The first example was the solution of the analysis problem for the irrotational flow through a channel representative of a vaneless diffuser for a supersonic compressor.

Prescribed conditions. - The outer wall of this channel was specified by the equation

$$- 2.2857z^{*2} - 0.49725r^{*2} + 8.6790z^{*} + 20.906r^{*} - 8.9930r^{*}z^{*} - 20.408 = 0 \tag{27}$$

between $r^{*} = 1.0$ and $r^{*} = 1.9351$. The inner contour was specified from $z^{*} = 0$ to $z^{*} = 0.59552$ by

$$z^{*2} + r^{*2} - 6.0600r^{*} + 3.9259 = 0 \tag{28}$$

and from $z^{*} = 0.59552$ to $r^{*} = 1.9351$ by

$$z^{*2} - 0.4721z^{*} + r^{*2} - 4.3050r^{*} + 2.7746 = 0 \tag{29}$$

The dimensionless inlet conditions were

Vortex constant, $r^{*}u^{*}$	0.8640
Axial velocity, w^{*}	1.3377
Radial velocity, v^{*}	0

Results. - The network for solution, or the points in the flow field of example I at which a solution was obtained, is shown in figure 5(a). Convergence of the ξ- or the η-lines in this figure indicate the formation of a weak shock. If the ξ- or η-lines intersect, a solution can not be obtained at these points since the solution is multivalued. The shocks occurring are considered to be weak and the assumption is made that conditions remain isentropic in this region. The dashed lines in figure 5(a) show the streamlines of the flow through the channel obtained from the values of ψ^{*} found during the solution. Figure 5(b) shows the contours of the resultant Mach number M throughout the channel which were obtained from the solution. This figure shows that the convergence of the characteristic lines running from $r^* = 1.07$, $z^* = 1.1$ on the inner wall to the point $r^* = 1.45$, $z^* = 1.23$ on the outer wall is accompanied by a slight compression wave, and the reflected compression wave of this system from the outer to the inner wall is preceded by an expansion wave (probably the result of the wall curvature) which shows up on figure 5(a) as a divergence of the characteristic lines. Also to be noted are the high outlet Mach numbers (which vary from 2.4 to 2.75) as compared with the inlet values (which vary from 2.3 to 2.9), despite the conversion of tangential kinetic energy into pressure. Figure 5(c) shows the contours of the meridional component of the Mach number throughout the channel; this particular component emphasizes the effect of the pressure waves, since these waves affect only the meridional velocity components. This figure brings out the particular compression wave cited in figure 5(a) and also the fact that the reflected compression is preceded by an expansion wave.

The contours of the tangential component of the Mach number through the channel are shown in figure 5(d). This figure has been included to complete the plot of Mach number components.

In figure 6 is shown the geometric relation between the α-coordinate used in figure 7 and the r^*- and z^*-coordinates. The α-coordinate is used for plotting the solution along the boundaries so that a comparison between the outer and inner contours can be made more readily. The distribution of the static pressure, the resultant velocity, the meridional velocity component along the outer and inner contours, and the area variations throughout the channel are shown in figure 7.

The variations of the resultant velocity and the pressure on the outer and inner boundaries correlate with each other as might be expected, but the average of the values on the two walls shows only a small over-all change from inlet to outlet. On the other hand, the meridional component shows a steady increase (except for an initial drop on the inner contour) in the region of constant area (fig. 7) except near the end, where the decreasing area tends to keep the average meridional component more nearly constant. The explanation of this anomalous behavior lies in the one-dimensional equation for mass flow

$$W = \rho A V_m \tag{30}$$

Differentiating this equation for the case of constant area after V_m has been expressed in terms of V and u yields

$$\frac{V\,dV - u\,du}{V_m^2} + \frac{d\rho}{\rho} = 0 \tag{31}$$

The differentials of the density ρ and the velocity V are expressed in terms of pressure by the Bernoulli equation

$$V\,dV = \frac{dp}{\rho} \tag{32}$$

and the isentropic relation

$$\frac{p}{\rho^{\gamma}} = \text{constant} \tag{33}$$

The differential of u is expressed in terms of r by use of the irrotationality condition

$$ru = \text{constant} \tag{34}$$

The result obtained is

$$\frac{d \ln p}{d \ln r} = \frac{\gamma M_{\theta}^2}{1 - M_m^2} \tag{35}$$

For increasing radius, diffusion will require that this derivative be positive; hence, M_m^2 must be less than 1.

Since for example I M_m is always greater than 1.0, an over-all decrease in pressure is to be expected from equation (35).

Another significant factor in figure 7 is the fluctuation of the resultant velocity V^*. The drop in velocity on the outer wall for $45^\circ < \alpha < 50^\circ$ shows the effect of compression waves originating on the inner wall at $35^\circ < \alpha < 40^\circ$ and gives an effect not calculable from simple one-dimensional flow or streamline flow with flow curvature estimated from wall curvature. There are similar important fluctuations elsewhere. In fact, the whole pattern of the resultant velocity from 40° to 67° on the inner wall appears to be a reflection of the outer wall pattern from 28° to 55°, and the trough on the outer wall from 47° to 60° is a reflection of the same pattern on the inner wall from 35° to 50°.

Example II

The second example is a solution of the design problem for the irrotational flow through a channel which might be representative of a transition channel between the outlet of a supersonic compressor and a vaned diffuser. Such a channel might be desirable from the standpoint of changing the velocity distribution leaving the outlet of a supersonic compressor to obtain a flow better adapted for inflow to the diffuser blades. This transition channel might also be used to accomplish part of the required diffusion by contracting it to the extent that the Kantrowitz contraction ratio is reached for the meridional component of the Mach number. (The Kantrowitz ratio is the ratio of inlet area to throat area when a normal shock at the inlet would cause choking at the throat.)

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Prescribed conditions. - The outer contour of the channel for this example was specified constant at a radius of 1.0 (dimensionless). The inlet conditions were the same as those specified in example I. The velocities could not be prescribed along the outer contour without overprescribing the problem until the point $(r^* = 1.0,\ z^* = 0.46$, fig. 8(a)) at which the characteristic originating at the inlet hub intersected the casing. From this point $(r^* = 1.0,\ z^* = 0.46)$ until the end of the channel, the velocity was prescribed by the following equation

$$w^* = 15.396z^{*3} - 26.812z^{*2} + 14.905z^* - 1.3437 \tag{36}$$

This equation corresponds to diffusion of the meridional Mach number down to the value of the Kantrowitz ratio for an infinitesimal stream tube along the outer wall at a passage length of $z^* = 0.70$. The other specified condition is that the radial component of velocity is zero at the outlet.

Results. - The solution of example II is shown graphically on figure 8. The propagation of effects along characteristics (fig. 8(a)) again indicates the inadequancy of the one-dimensional or streamline curvature methods of computation. Between $\psi^* = 0$ and $\psi^* = 0.2$, for example, a throat exists at $z^* \sim 0.44$. A similar minimum is seen between other streamlines. These minimums are located on the characteristic line between the points $z^* = 0.375$, $r^* = 0.76$ to $z^* = 0.70$, $r^* = 1.00$. Convergence and divergence of the characteristics indicate compression and expansion, respectively, as may be seen by comparison of figures 8(a) and 8(c).

The constant velocity on the outer contour from $z^* = 0$ to $z^* = 0.46$ (fig. 8(e)) is a consequence of the dependence of the velocity there on the inlet conditions (see characteristic curves, fig. 8(a)). (The possibility of a short diffuser of this type is thus eliminated.) The prescribed velocity drop from this point on was attained by compression waves formed by the calculated curvature of the inner contour from $z^* = 0$ to $z^* = 0.375$ (fig. 8(a)). The result is a pressure rise on the inner contour in the same region (fig. 8(e)). The outlet conditions of $v^* = 0$ demand a reflex in this curvature and a pressure drop in the following region.

Example III

Prescribed conditions. - The third example is similar to the second with the following differences:

(1) The inlet tangential velocity was prescribed constant instead of inversely proportional to the radius and equal to 0.8640. The flow is therefore rotational; whereas, for cases I and II the flow is irrotational. The inlet velocity vector at the outer contour was the same as that of example II, and likewise, the radial velocity was zero.

(2) The meridional velocity at the outer contour was prescribed after the intersection point with the characteristic line originating at the inner contour and the inlet. The value of w^* in this region was:

$$w^* = 22.1308z^{*3} - 39.4467z^{*2} + 22.6930z^* - 2.9145 \qquad (37)$$

The initial and final values for w^* in this region are the same in examples II and III.

Results. - Because the rotational velocity is lower near the inner contour than for example II, the axial velocity components are higher and most of the differences between the two examples are a result of this velocity difference. The results are summarized in figure 9. Because of the expected decrease in Mach angles shown in figure 9(a), this diffuser is inherently longer than that for example II, but the diffusion rate at the outer contour was increased to maintain the same over-all axial length. All the effects noted in example II may be seen in example III with the modification for smaller values of ζ (characteristic angles). The shortening of the region of diffusion on the inner contour and the shifting of the velocity rise region to the left, for example, may thus be explained from the change in the characteristic directions.

SUMMARY OF RESULTS

The axisymmetric flow between two surfaces of revolution with vorticity and tangential velocity was set up for solution by the method of characteristics. The criterion for the existence of characteristics is that the meridional component of the velocity be greater than sonic. Although the flow configuration is apparently simple, a hodograph solution similar to that for plane flows is not available because of the nonhomogeneity of the equations and because the coefficients of the differential equation are functions of position as well as of velocity. Some of the main results of this work are contained in the following statements:

1. In the numerical example of flow between two highly curved surfaces of revolution, pressure waves arose which impinged on the opposing wall and reflected. Pressure variations therefore have a period structure and a similarity will exist on the two boundaries.

2. A one-dimensional analysis showed that in the case in which along the flow path the flow area is constant and the radius increases so that the tangential component of velocity decreases, there will be a diffusion process only if $M_m^2 < 1.0$, where M_m is the meridional component of the local Mach number. This fact was confirmed by the two-dimensional analysis of the example in which there was a radial flow.

3. In all examples computed, the pattern of convergence or divergence of the characteristic curves indicated compression or expansion waves which were the same as would be expected in two-dimensional plane flow.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, May 14, 1952

APPENDIX - SYMBOLS

The following symbols are used in this report:

A	area
a	sonic velocity of gas
g	function of z (see equation (25))
H	enthalpy
i	unit vector parallel to curve for θ = constant and z = constant ($i = \nabla r$)
j	unit vector parallel to curve for z = constant and r = constant ($j = r\nabla\theta$)
k	unit vector parallel to curve for r = constant and θ = constant ($k = \nabla z$)
M	local Mach number
P,P'	points defined by figures 2 and 3
p	static pressure of gas
r	normal distance from axis of symmetry
u	rotational component of gas velocity
V	resultant gas velocity ($\overline{V} = iv + ju + kw$)
v	radial component of gas velocity
w	axial component of gas velocity
X_+,X_-	abbreviations (see equation (16))
Y_+,Y_-	abbreviations (see equation (17))
z	distance measured parallel to axis of symmetry
α	angular position coordinate (see fig. 6)
γ	ratio of specific heats

ζ_+ slope of curve for ξ variable and η constant

ζ_- slope of curve for η variable and ξ constant

η one of the characteristic coordinates

θ meridional angle (for cylindrical coordinates) 2499

ξ one of the characteristic coordinates

ρ gas density

σ either η or ξ in general form of equations

ψ stream function $(\psi = \rho r j X \overline{V})$

Subscripts:

i inlet

m meridional component

r,z,σ partial differentiation with respect to variable indicated by subscript

T stagnation conditions

θ tangential component

1,2,3,4 point or line positions (see figs. 2 to 4)

Superscript:

* dimensionless quantity

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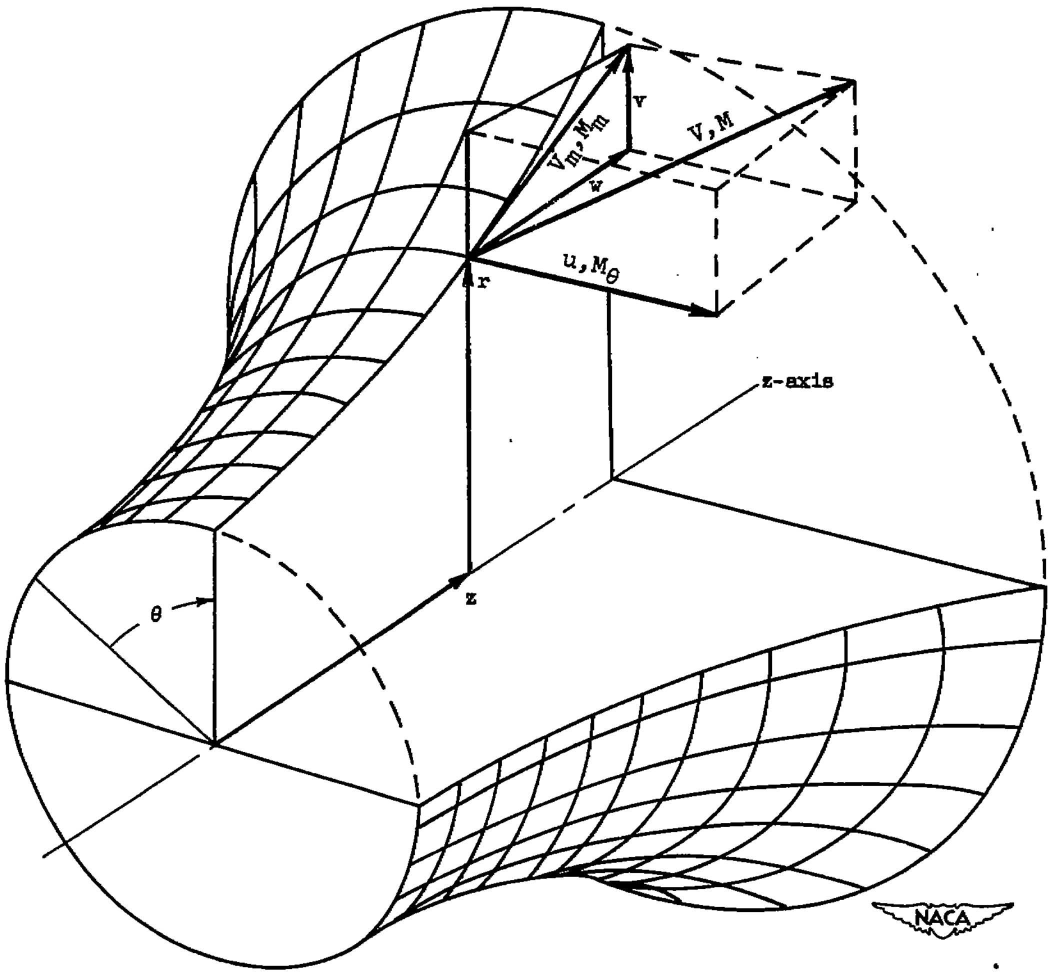

Figure 1. - Curved stream surface of revolution showing coordinates and velocity components.

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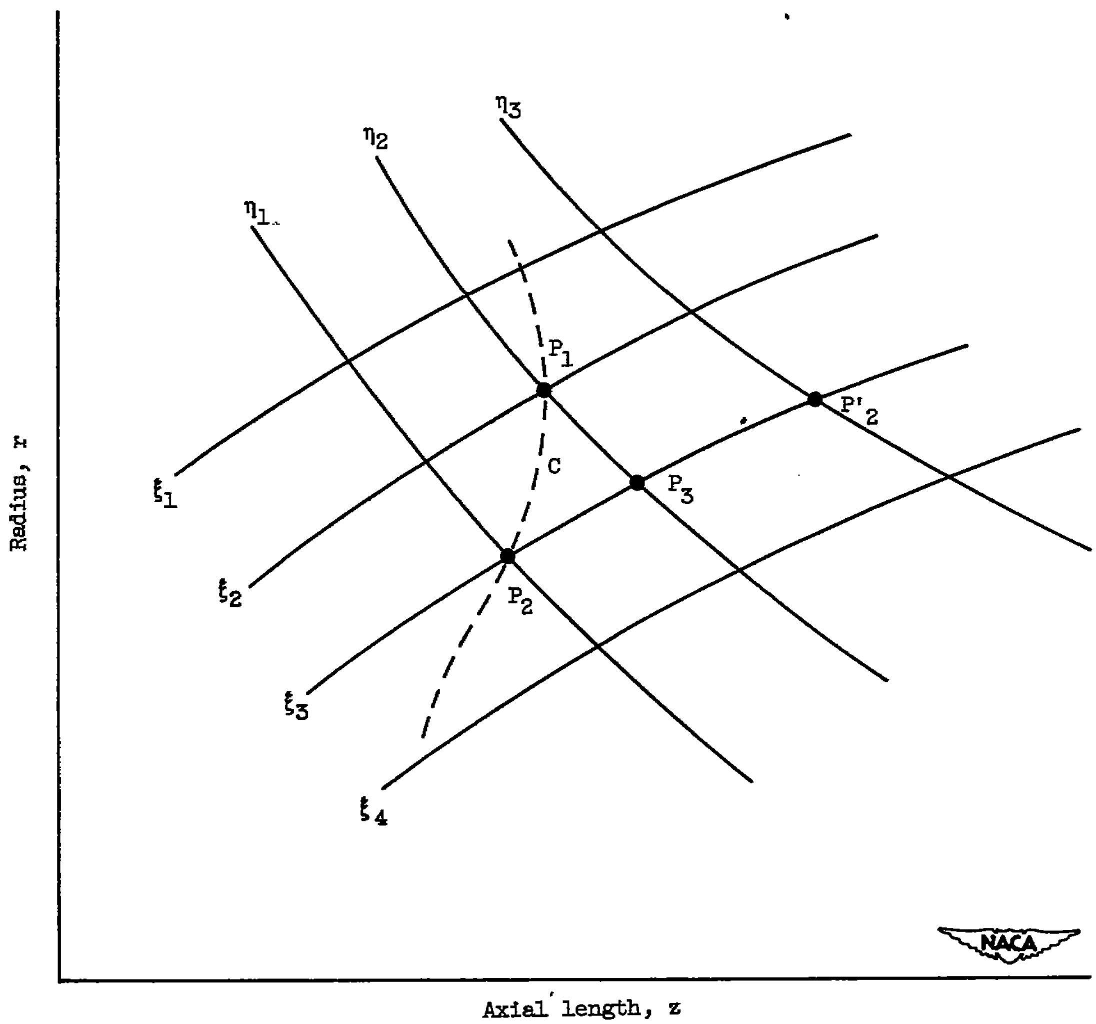


Figure 2. - Characteristic and cylindrical coordinates.

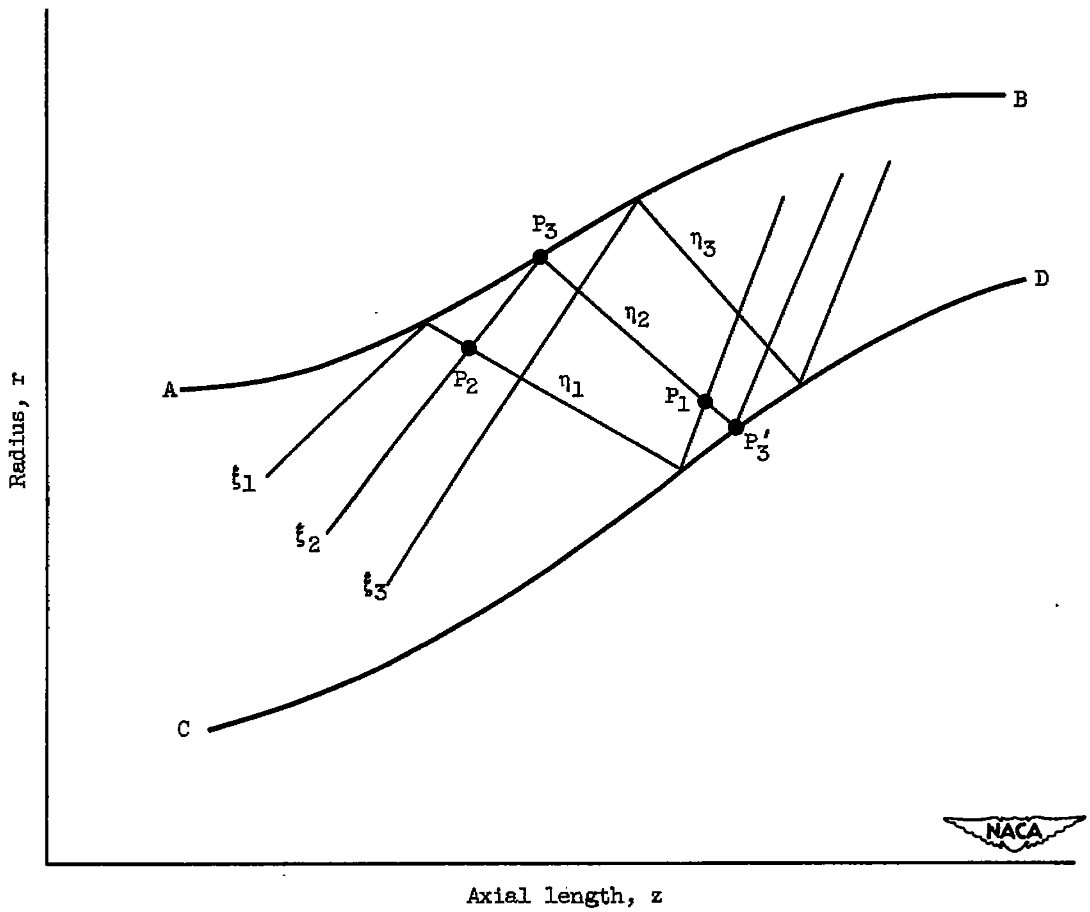


Figure 3. - Sketch to illustrate discussion of relations between characteristic lines and boundaries for supersonic analysis problem in annular channel.

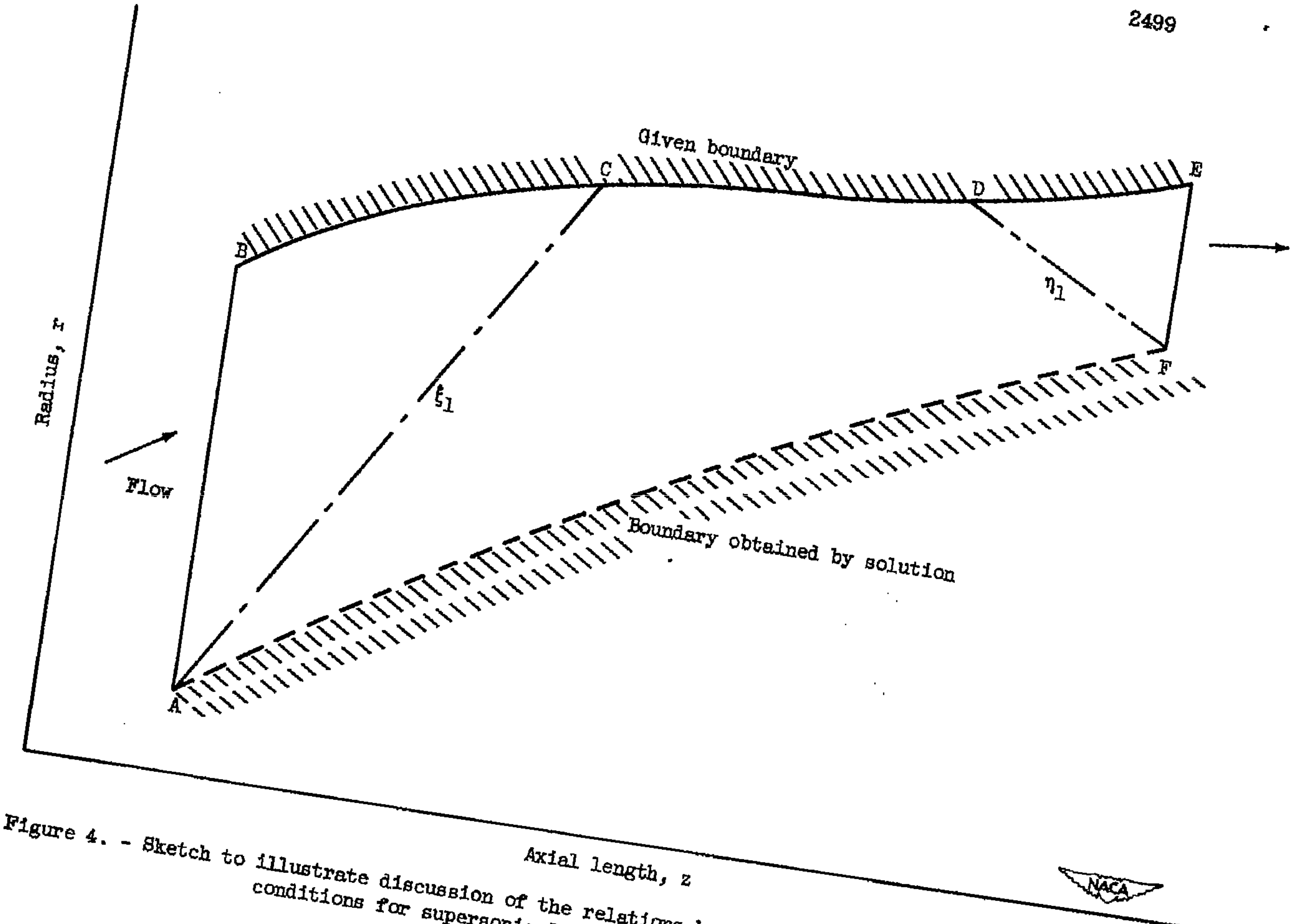
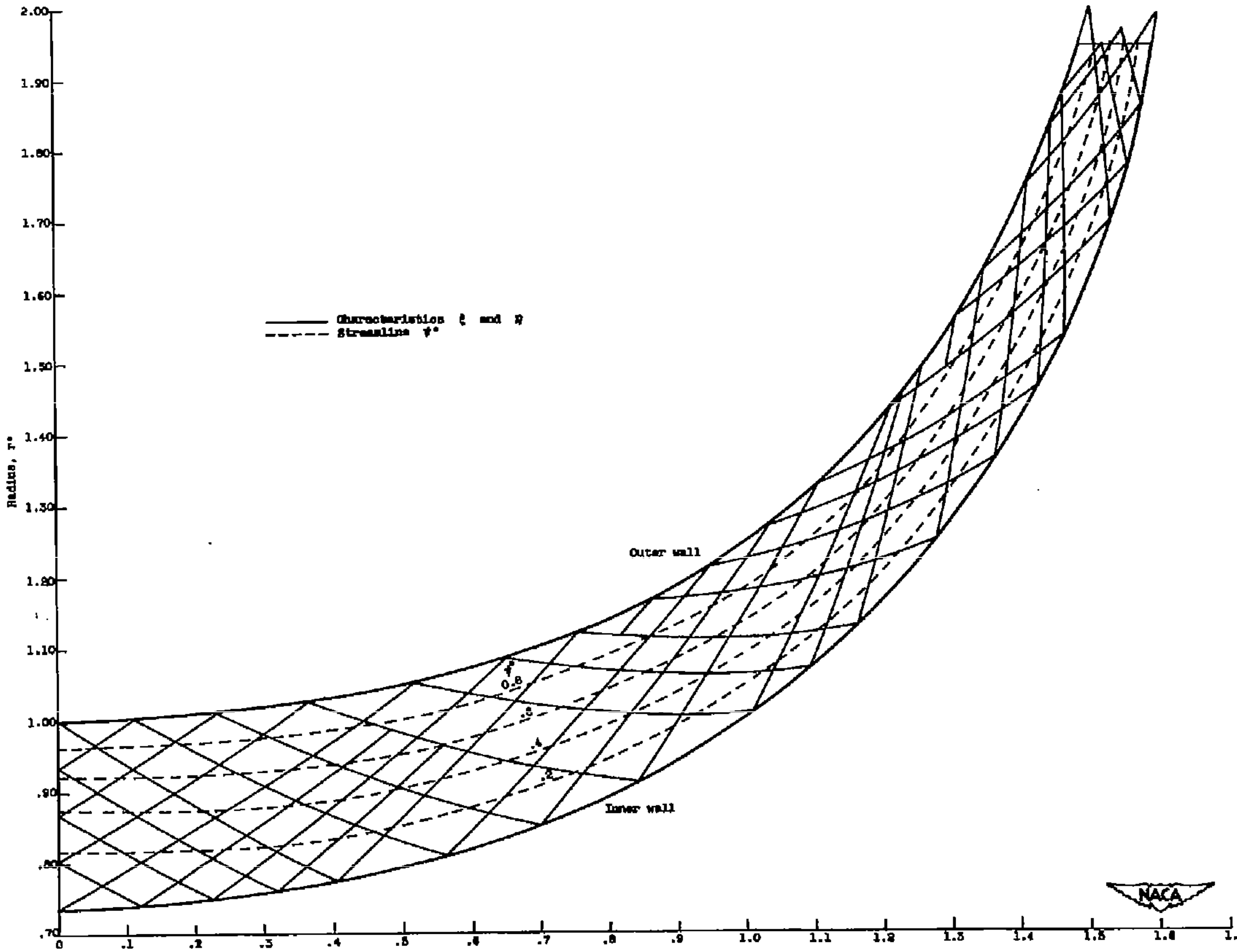
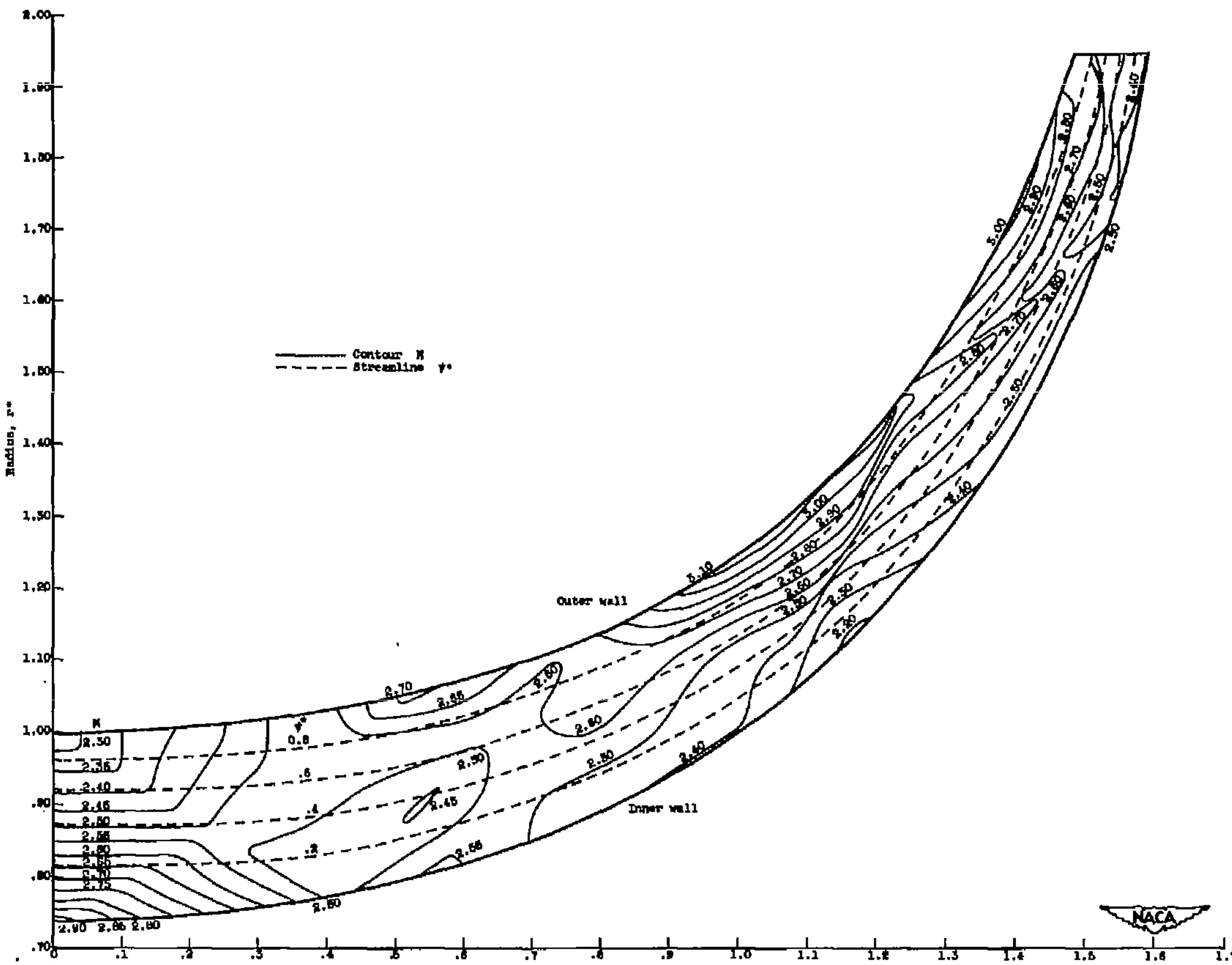


Figure 4. - Sketch to illustrate discussion of the relations between boundary and specified boundary conditions for supersonic design problem in annular channel.



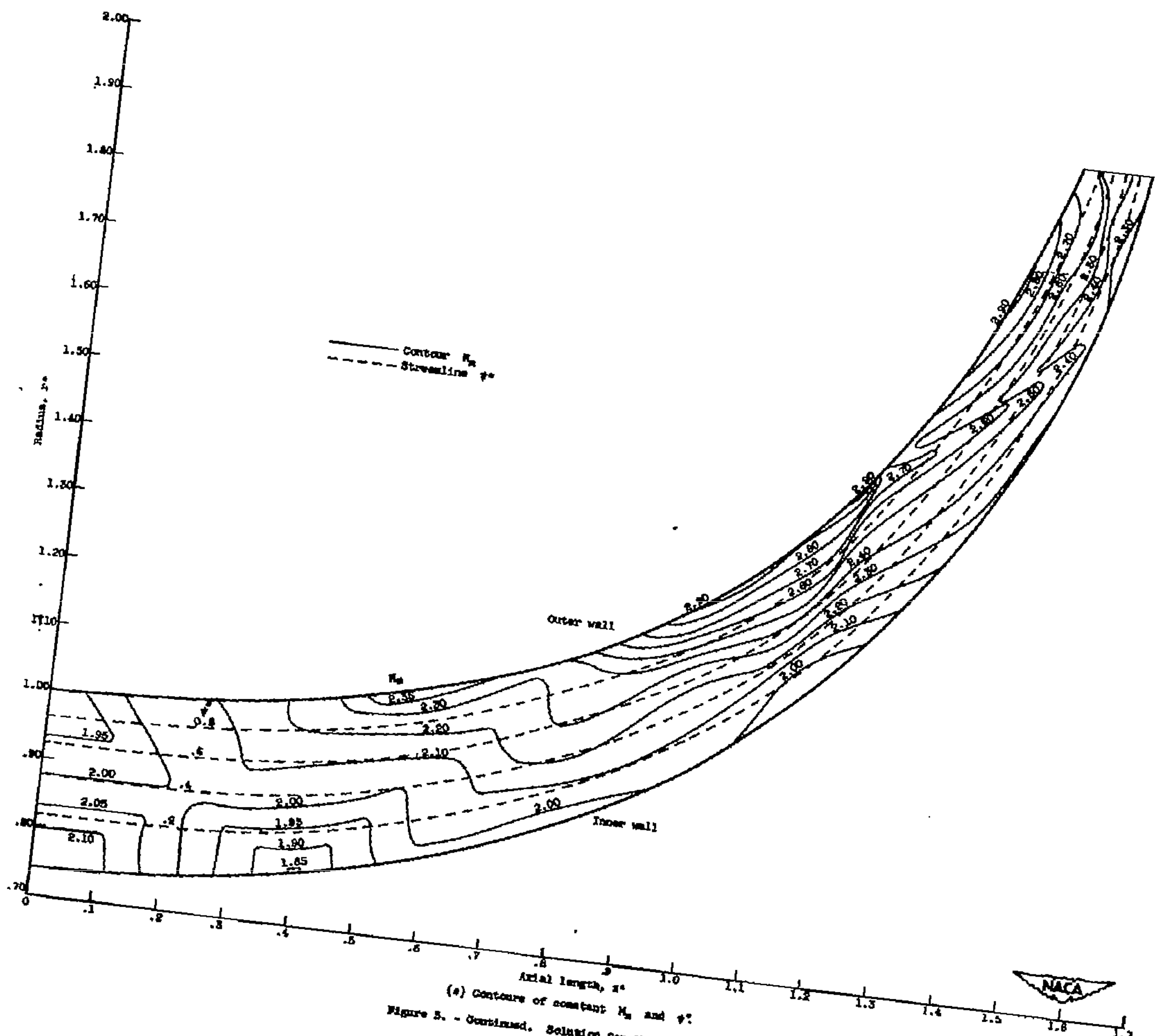
(a) Contours of constant ξ, η, and ψ*.

Figure 5. - Solution for flow in example I.



(b) Contours of constant M and ψ*.

Figure 5. - Continued. Solution for flow in example I.



(a) Contours of constant M_R and ψ^*.

Figure 5. - Continued. Solution for flow in example I.

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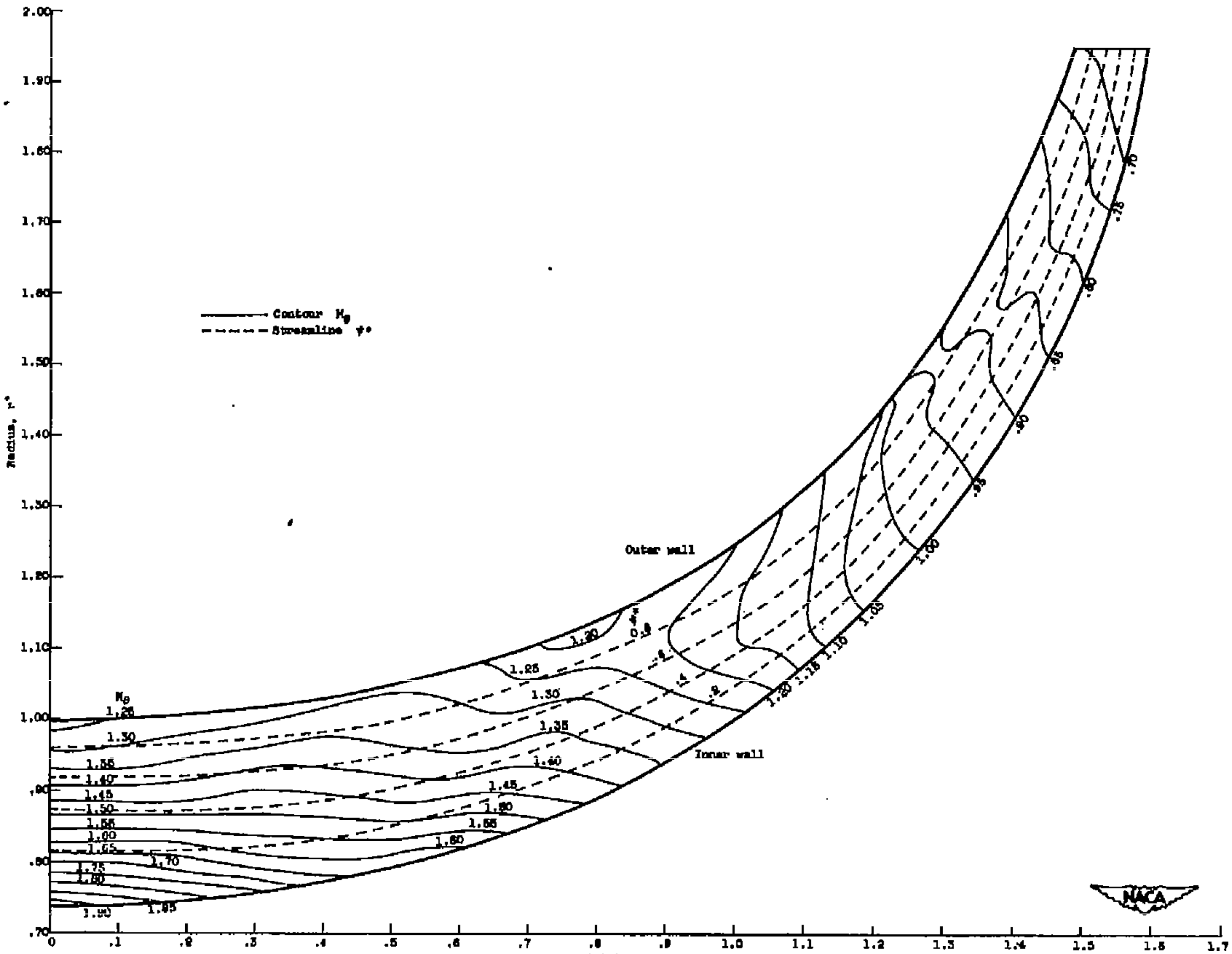

(d) Contours of constant M_θ and ψ^*.

Figure 5. - Concluded. Solution for flow in example I.

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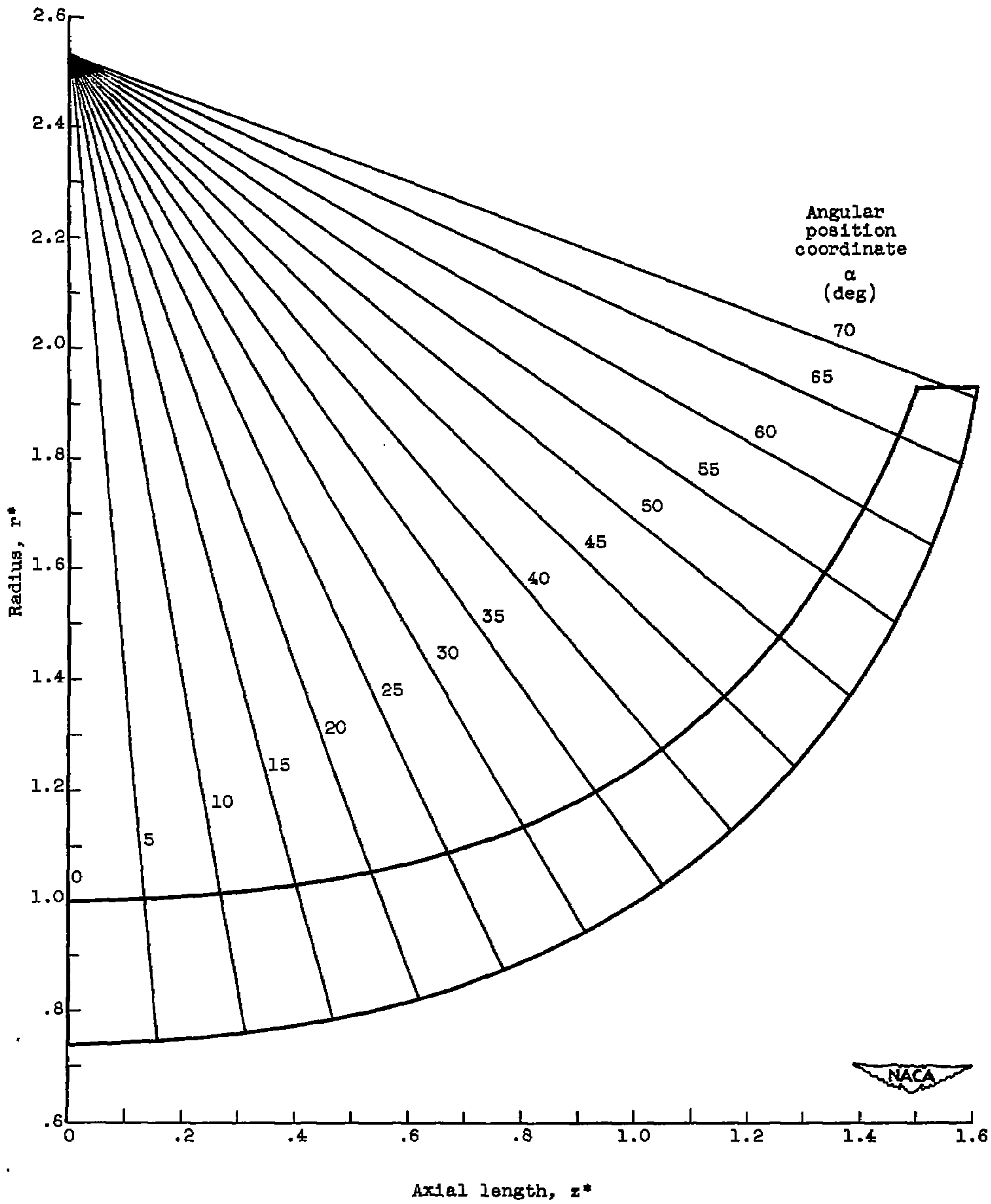


Figure 6. - Relation between diffuser coordinates r*, z*, and α for example I.

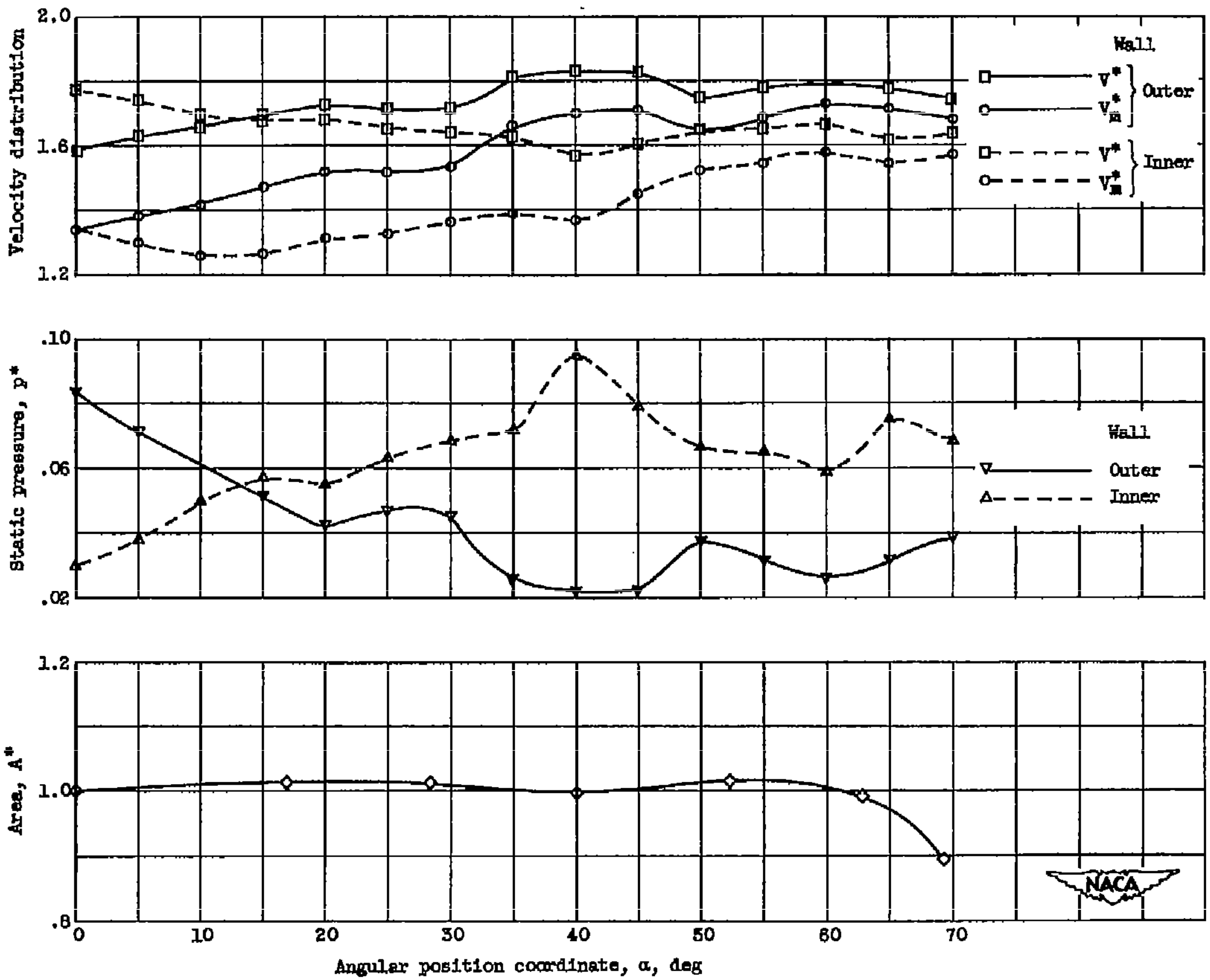
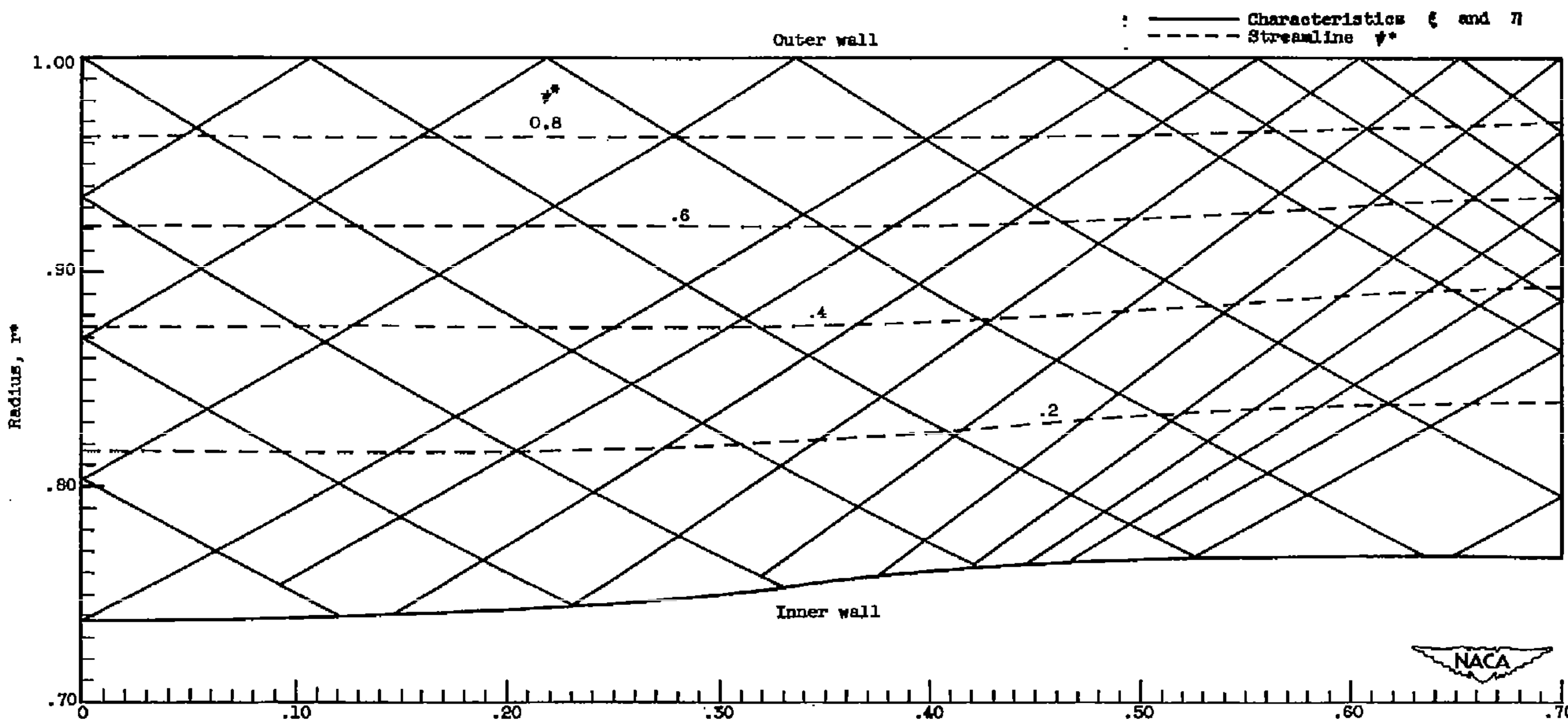
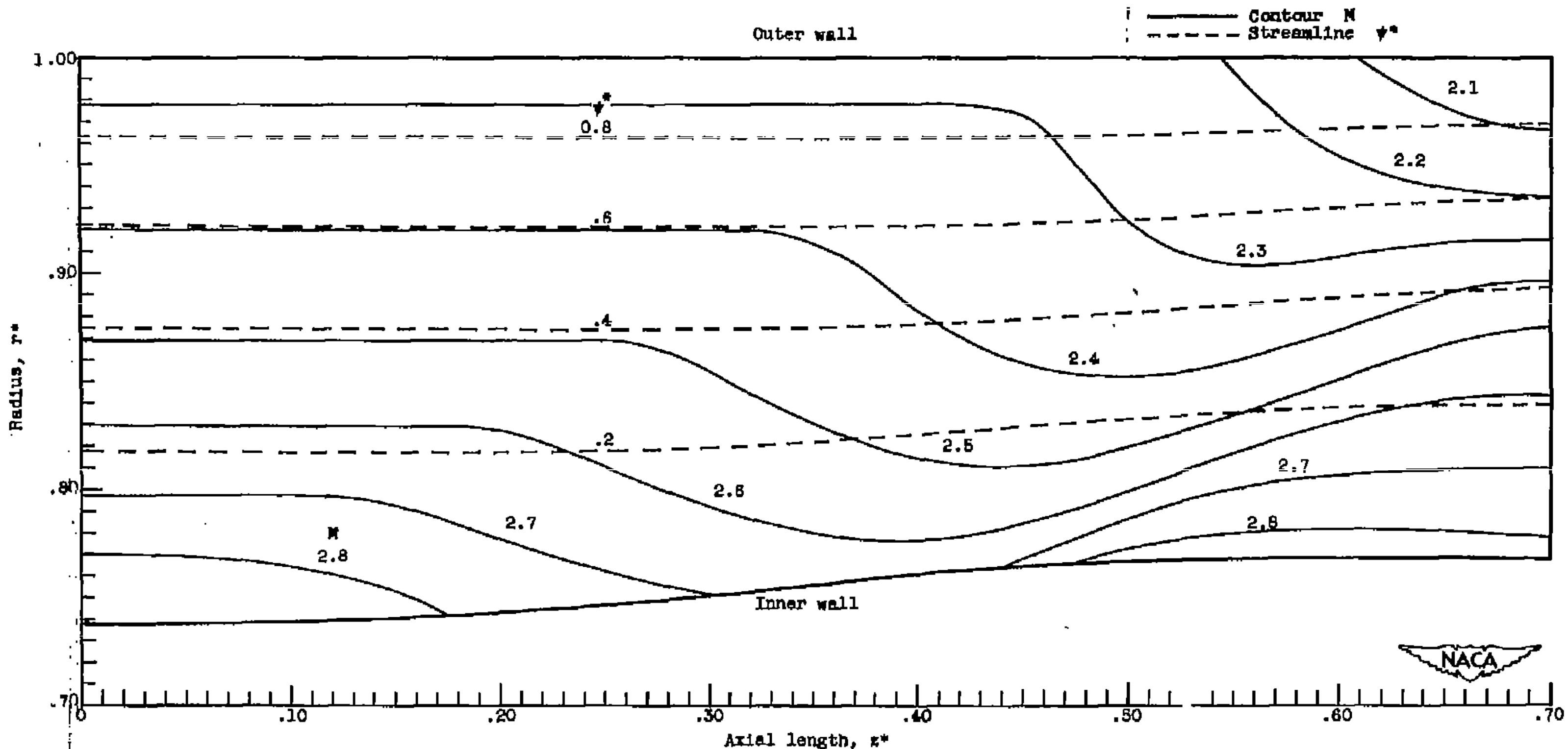


Figure 7. - Velocities and pressure on outer and inner walls and flow area through annular channel for example I.



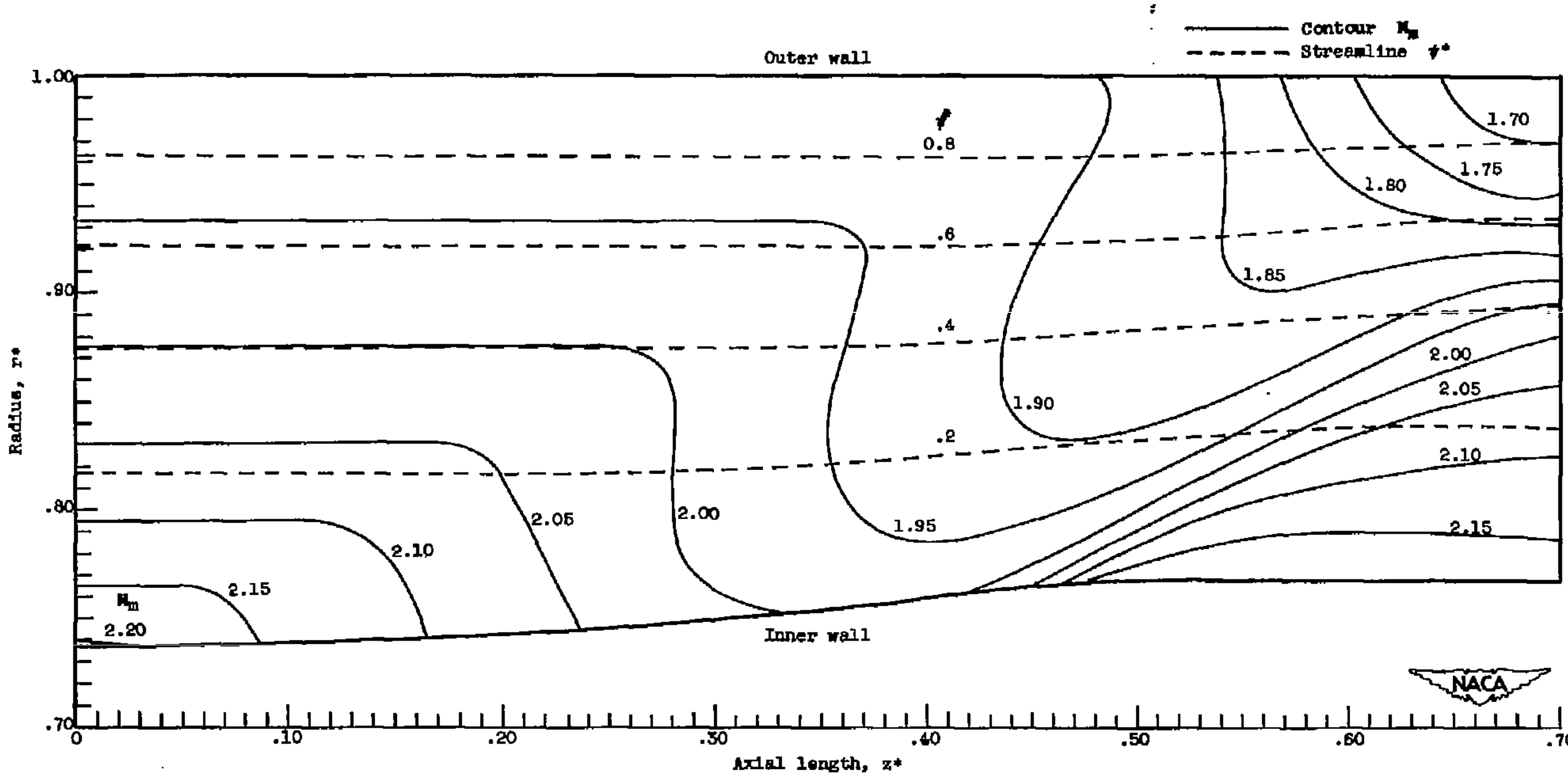
(a) Contours of constant ξ, η, and ψ^*.

Figure 8. - Solution for flow in example II.



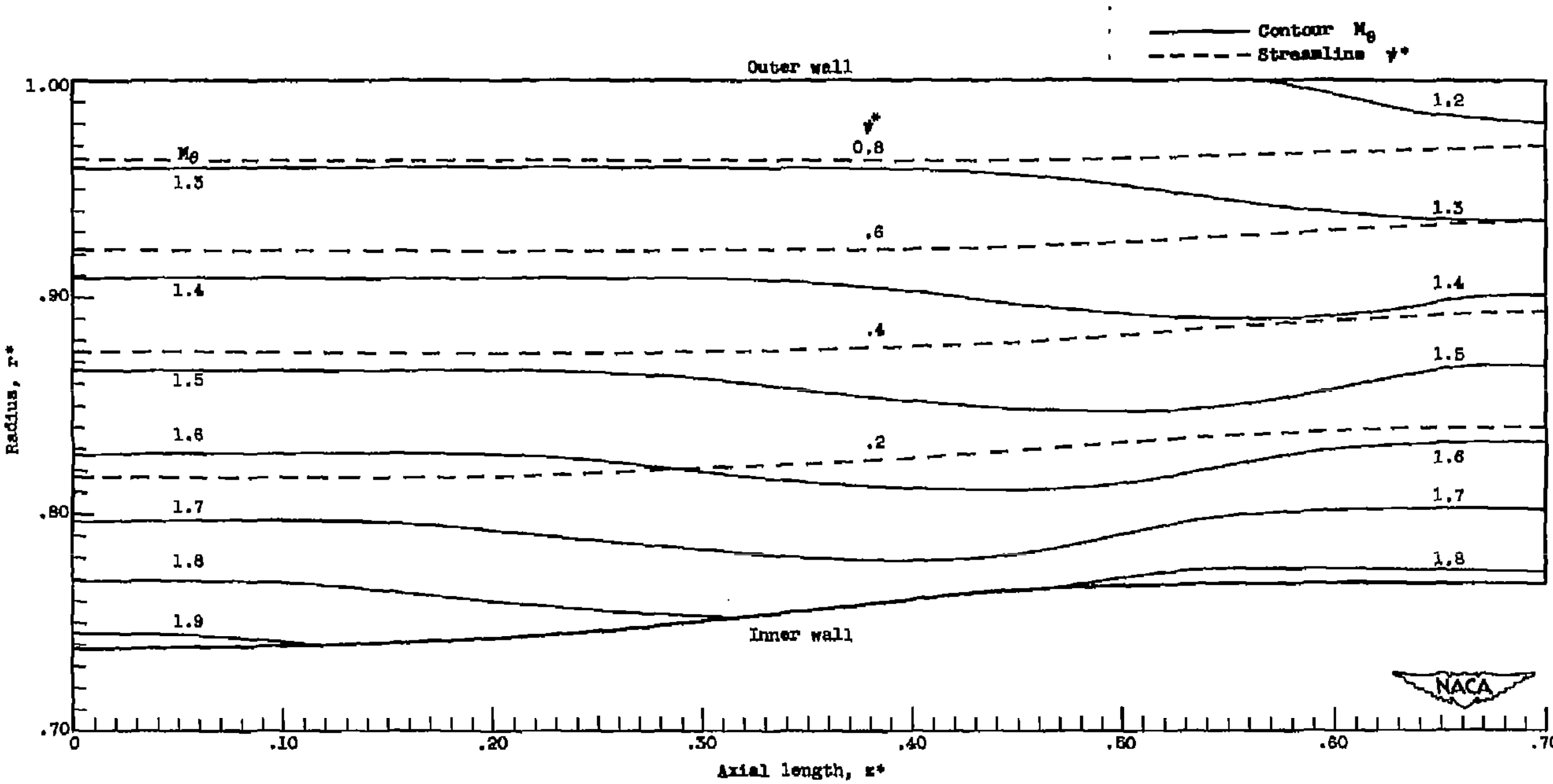
(b) Contours of constant M and ψ*.

Figure 8. - Continued. Solution for flow in example II.



(c) Contours of constant M_m and ψ^*.

Figure 8. - Continued. Solution for flow in example II.



(d) Contours of constant M_θ and ψ^*.

Figure 8. - Continued. Solution for flow in example II.

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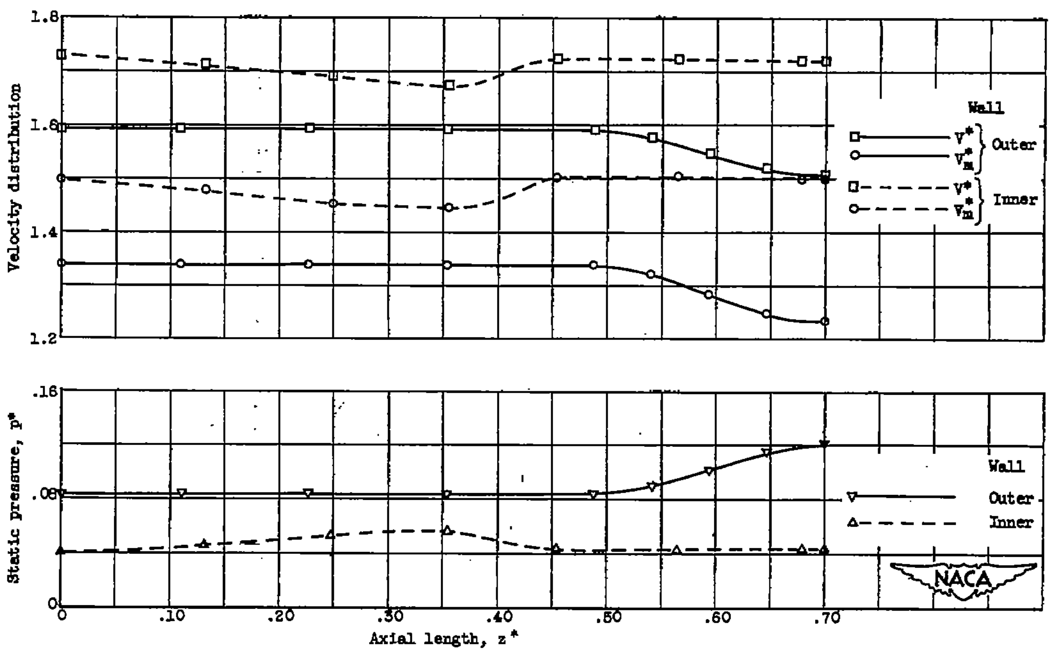
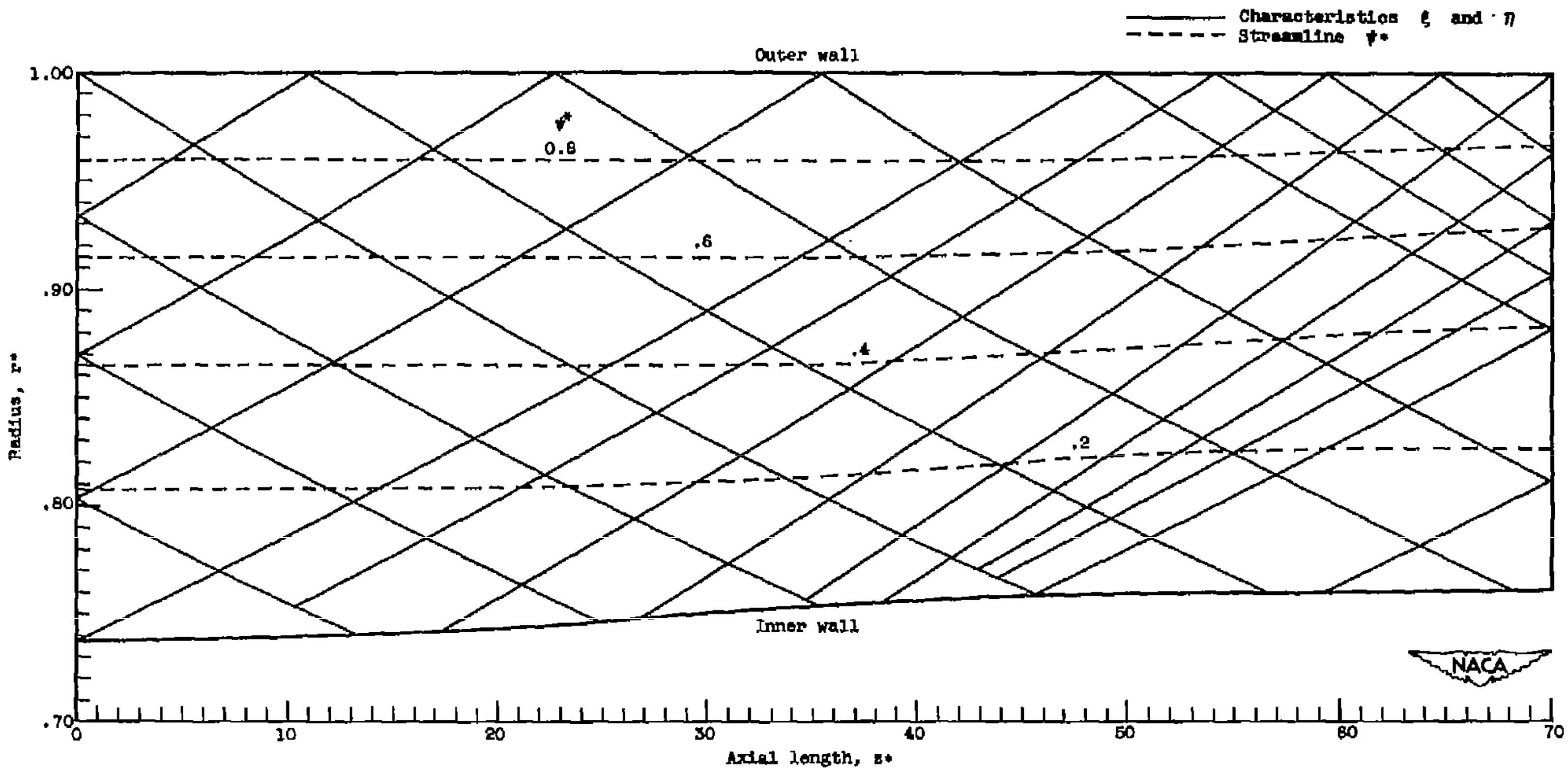


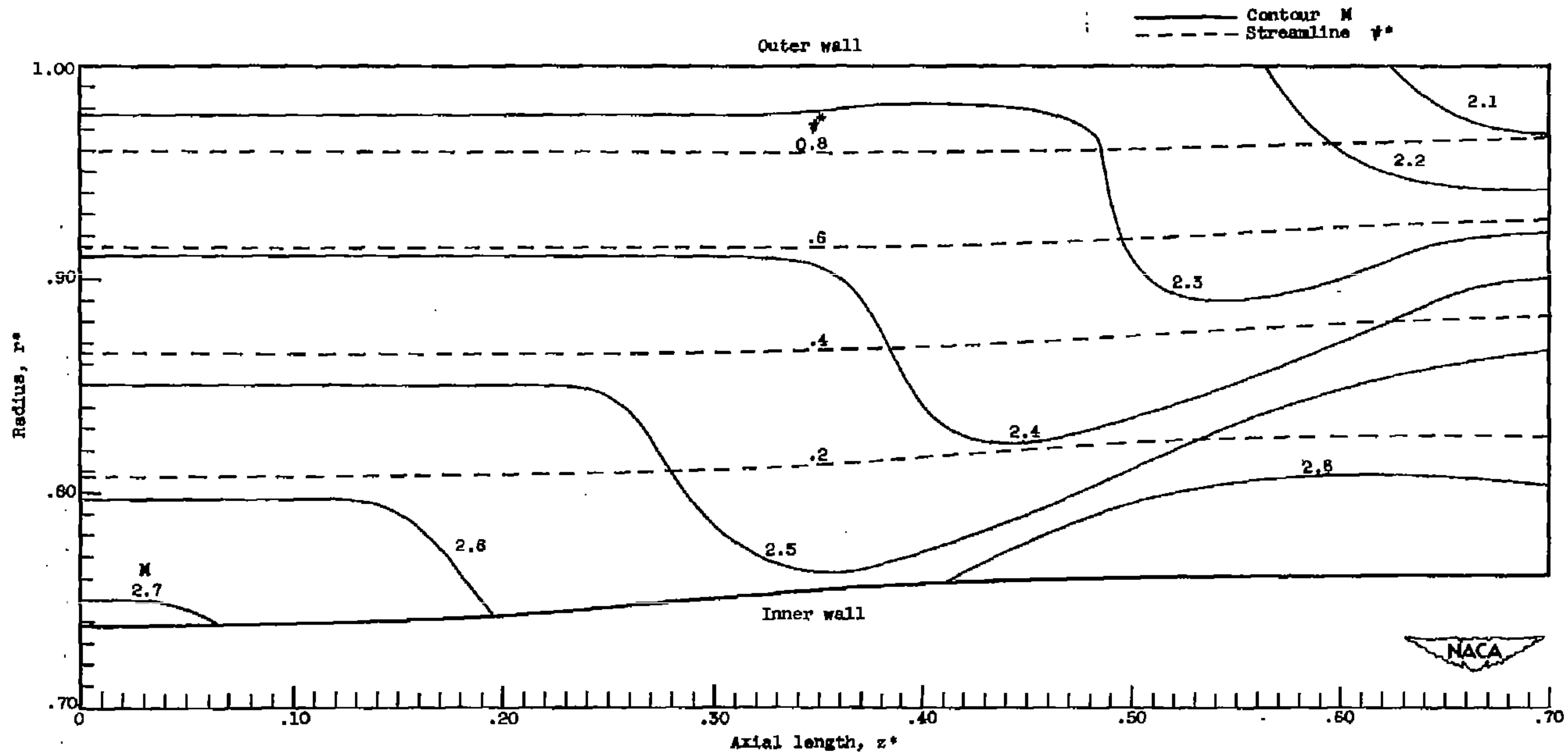
(e) Velocities and pressures on outer and inner walls.

Figure 8. - Concluded. Solution for flow in example II.



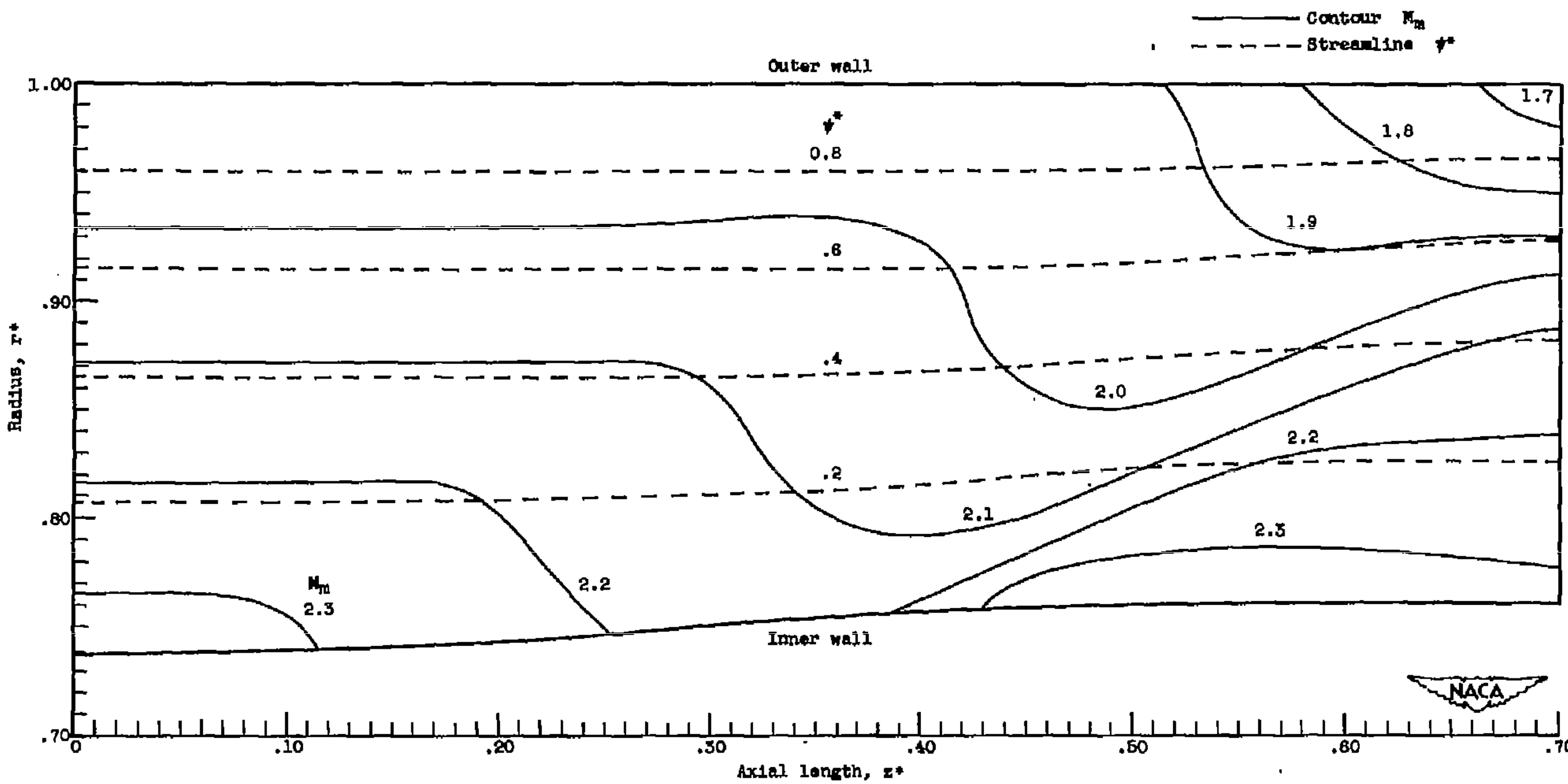
(a) Contours of constant ξ, η, and ψ*.

Figure 9. - Solution for flow in example III.



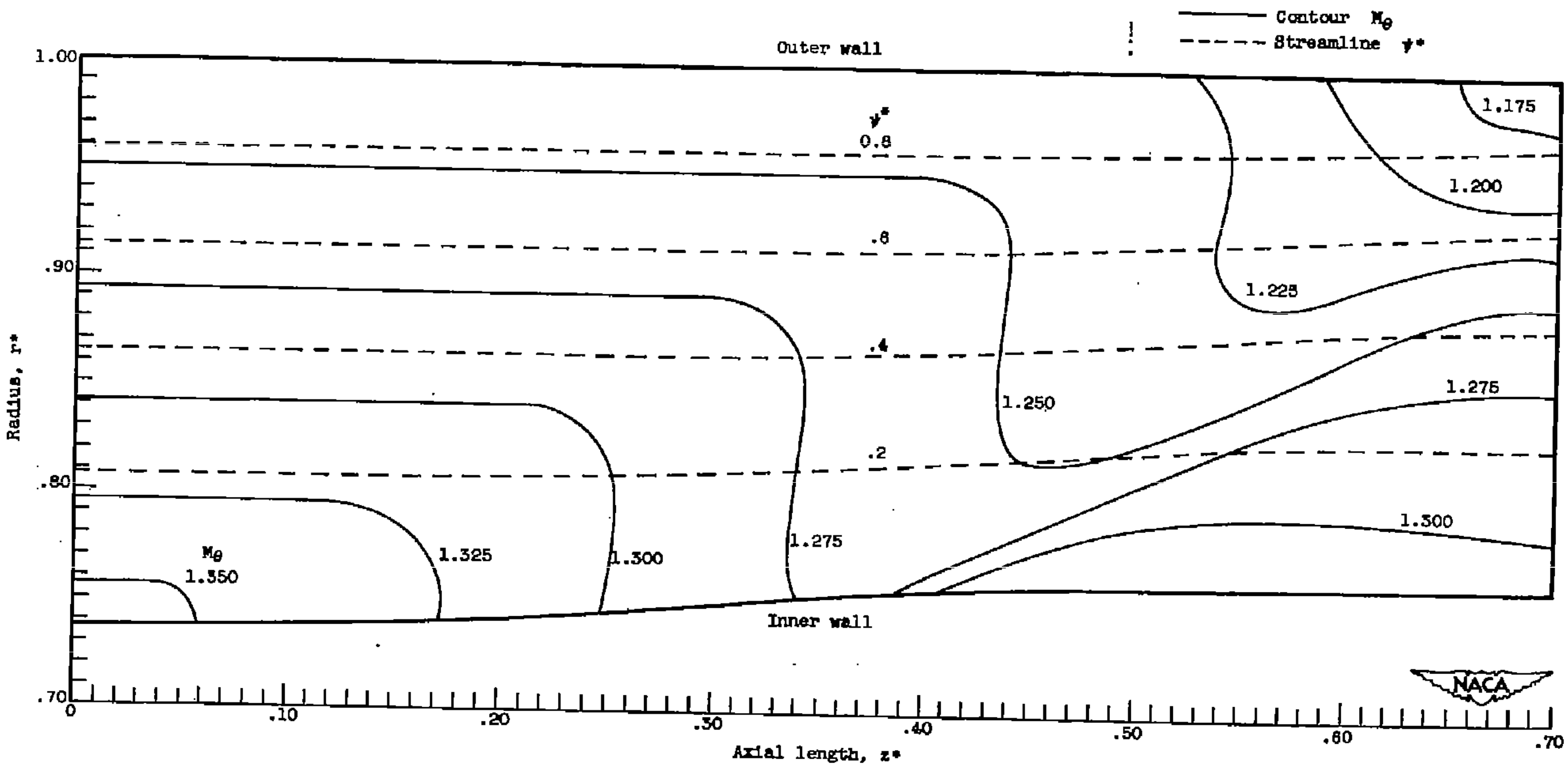
(b) Contours of constant M and ψ*.

Figure 9. - Continued. Solution for flow in example III.



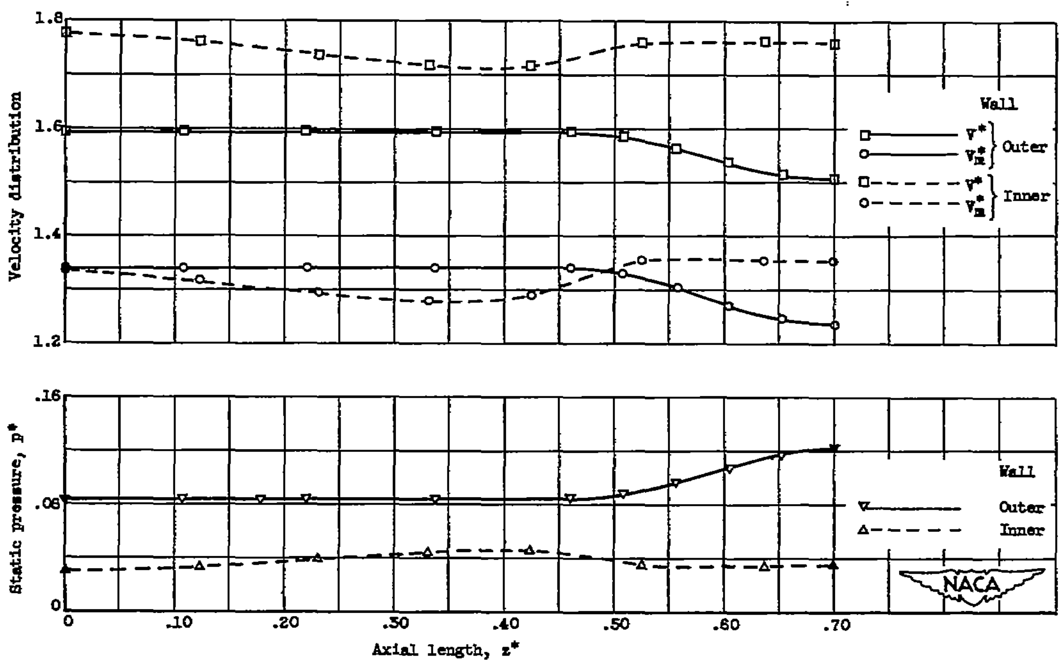
(c) Contours of constant M_m and ψ^*.

Figure 9. - Continued. Solution for flow in example III.



(d) Contours of constant M_θ and ψ^*.

Figure 9. - Continued. Solution for flow in example III.



(e) Velocities and pressures on outer and inner walls.

Figure 9. - Concluded. Solution for flow in example III.

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